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NEWTON LIKE - MINIMAL RESIDUAL METHODS APPLIED TO TRANSONIC FLOW CALCULATIONS

Yau Shu Wong

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INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

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NEWTON LIKE - MINIMAL RESIDUAL METHODS APPLIED TO TRANSONIC FLOW CALCULATIONS

Yau Shu Wong
Institute for Computer Applications in Science and Engineering
and
McGill University, Montreal, Quebec

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Abstract

A new computational technique for the solution of the full potential equation is presented. The method consists of outer and inner iterations. The outer iterate is based on a Newton like algorithm, and a preconditioned Minimal Residual method is used to seek an approximate solution of the system of linear equations arising at each inner iterate. The present iterative scheme is formulated so that the uncertainties and difficulties associated with many iterative techniques, namely the requirements of acceleration parameters and the treatment of additional boundary conditions for the intermediate variables, are eliminated. Numerical experiments based on the new method for transonic potential flows around NACA 0012 airfoil at different Mach numbers and different angles of attack are presented, and these results are compared with those obtained by the Approximate Factorization technique. Extention to three-dimensional flow calculations and application in finite element methods for fluid dynamics problems by the present method are also discussed.

I. Introduction

The ability to compute transonic flow fields around airfoils or wings is an important aid in the design of efficient modern transport aircrafts since they operate predominantly in transonic ranges. Considerable effort has been spent, in recent years, on the construction of fast and accurate numerical procedures for the solution of the full potential equation. To be useful as a design and analysis tool, the success of a computational

procedure should not be problem dependent. For example, some numerical procedures yield rapidly converged solutions if optimal values of acceleration parameters are provided and if other special conditions are given. However, it should be pointed out that optimal values of these parameters are often unobtainable for practical calculations.

The standard iterative procedure for transonic small perturbation and full potential calculations was based on the successive line over-relaxation (SLOR) method. Because of its slow convergence rates for many practical problems, the method has been replaced by many new iterative procedures. One of the most successful numerical techniques is based on the Approximate Factorization (AF) scheme, and there are many variants of the AF method[1-6] including those based on ADI[1] type developed by Ballhaus and Steger, AF2[2] type by Ballhaus, et al., AF3[6] type by Baker, and SIP[5] type by Sankar, et al. These computational procedures provide substantial improvement in rates of convergence compared to the SLOR method. However, they all require one or more iteration parameters in order to accelerate the convergence, and an intermediate variable is also introduced into the iterative process for a two-dimensional calculation. Consequently, the uncertainty as to what values should be used for the iteration parameters, and the uncertainty about how to select the boundary conditions for the intermediate variable, may affect the convergence rates as well as the stability of the iterative process. It is our aim here to present an efficient iterative procedure which yields a rapid rate of convergence like the AF scheme, while eliminating the difficulties associated with the AF scheme. The present method consists of outer and inner

iterations. The outer iterate is based on a Newton like iterative process in which the Jacobian matrix is not required, and a preconditioned Minimal Residual algorithm is applied only to seek an approximate solution of the system of linear equations arising at each inner iterate. This method can therefore be regarded as a Newton like - Minimal Residual algorithm or an Inexact Newton like (IN) iterative procedure.

The idea of the IN iterative scheme was first proposed by the author in [7]. Although our early paper indicated that the method can be used to compute transonic flow fields around airfoils, it was not competitive with the AF scheme implemented by Dougherty et al[8]. The computational results showed that more iterations were needed for a converged solution compared to the AF scheme, and the CPU time per iteration for the IN method was about three times that required for the AF scheme. More recently, the IN method has been modified to include a better preconditioning operator so that a substantial improvement has been achieved: the number of iterations is now about half of that required by the AF scheme, and the CPU time per iteration is about twice of that required by the AF scheme. The present paper is mainly concerned with the numerical solutions of a two-dimensional full potential equation, and particular attention is focused on the improved version of the IN iterative method. Comparisons of numerical results for lifting and non-lifting airfoil calculations between the IN and AF schemes are given and extension to three-dimensional calculations will be discussed. We describe the problem formulation for the transonic flow fields around an airfoil in section II, and the solution of nonlinear discrete potential equations by the IN method is presented in

section III, results of computational experiments are discussed in section IV, and finally, concluding remarks are given in section V.

II. Problem Formulation

For a two-dimensional problem in Cartesian coordinates, the governing partial differential equation for an inviscid isentropic fluid flow expressed in the conservation form is

$$(\rho\phi_x)_x + (\rho\phi_y)_y = 0 \tag{1}$$

where

$$\rho = \left[1 - \frac{\gamma - 1}{\gamma + 1} (\phi_x^2 + \phi_y^2) \right]^{1/(\gamma-1)}$$

Equation (1) is known as the full potential equation, where ϕ is the velocity potential, ρ, the density of the fluid flow, and γ, the ratio of specific heats. Equation (1) is a nonlinear equation since ρ is a function of ϕ_x and ϕ_y. The numerical solution of Equation (1) for transonic flow is more delicate and more interesting than those for purely subsonic or purely supersonic, because the governing equation changes its type from elliptic in subsoinc regions to hyperbolic type in supersonic regions, and the boundary between these regions is unknown. Moreover, the equation also admits discontinuous solutions, such as shocks which may exist in the flow fields.

To handle a general flow problem with complex geometries it is advantageous to transform Equation (1) from the physical domain in the Cartesian coordinates into the computational domain in a rectangle[8]. The full potential equation written in the

computational coordinates ξ and η is given by

$$\left(\frac{\rho U}{J}\right)_{\xi} + \left(\frac{\rho V}{J}\right)_{\eta} = 0 \qquad (2)$$

where

$$\rho = \left[1 - \frac{\gamma-1}{\gamma+1} \left(U\phi_{\xi} + V\phi_{\eta}\right) \right]^{1/(\gamma-1)},$$

$$U = A_1 \phi_{\xi} + A_2 \phi_{\eta}, \quad V = A_2 \phi_{\xi} + A_3 \phi_{\eta},$$

$$J = \xi_x \eta_y - \xi_y \eta_x, \quad A_1 = \xi_x^2 + \xi_y^2,$$

$$A_2 = \xi_x \eta_x - \xi_y \eta_y, \quad A_3 = \eta_x^2 + \eta_y^2 .$$

Here U and V are the contravariant velocity components along the ξ and η directions, J is the Jacobian of the coordinate transformation, A_1, A_2 and A_3 are metric quantities.

One of the difficulties in the numerical solution of transonic flow calculations is that both compression and expansion shocks are admitted by Equation (1). The expansion shocks, however, are physically meaningless. Thus in order to eliminate the expansion shocks from the flow fields, an artificial viscosity term is introduced, via an upwind bias, into the full potential equation. In this paper, the method of artificial density[8-9] is implemented, where the fluid density is modified in such a way that

$$\rho \leftarrow (\rho - \mu\rho_{\xi} \Delta\xi) \qquad (3)$$

where

$$\mu = \max [\, 0, \; 1 - 1/M^2 \,]$$

Here s $\leftarrow$ (t) indicates that s is replaced by t. In the above expression μ is a switching function which is zero in subsonic

flow fields and non-zero in supersonic flow fields, M is the local Mach number, and ρ_ξ is the density gradient in the upwind diretion. An important advantage in using the artificial density method is that a central difference approximation can be employed to discretize the full potential equation in the entire flow fields regardless of whether it is in a subsonic or a supersonic region.

III. Solution Procedure

By the application of the finite difference method the solution of the full potential equation (2) is transformed to the solution of a large set of nonlinear equations

$$L(\phi) = 0 \tag{4}$$

where ϕ is a vector of velocity potential at the grid points, and L is the nonlinear full potential operator.

Newton Like Algorithm

Our iterative scheme for the solution of Equations (4) can be described as follows:

Let ϕ° be an initial guess for the velocity potential vector, compute the residual vector $r^\circ = L(\phi^\circ)$, then for n=0,1,2,..., until $||r^n||_2 < \epsilon$,

$$\begin{aligned} &\text{solve} && M_n \delta\phi^n = -r^n \\ &\text{set} && \phi^{n+1} = \phi^n + \delta\phi^n \\ &\text{compute} && r^{n+1} = L(\phi^{n+1}) \end{aligned} \tag{5}$$

where n is an iteration number, $\delta\phi$ is the correction vector, and M_n is a matrix operator which varies from iteration to iteration. It should be noted that if $M_n = L'(\phi^n)$, the Jacobian matrix of

$L(\phi)$, then (5) is a Newton's iterative process for the solution of the nonlinear equations (4). Although Newton's method yields a rapid convergence rate, the method requires the initial guess ϕ° to be inside a domain of attraction, that is, one must have a good initial vector to ensure for convergence. Furthermore, even if the linearized full potential operator is a sparse matrix, the Jacobian matrix $L'(\phi)$ will likely be a full matrix. For many practical problems in aerospace industry the order of the nonlinear equations is large, such as 5000 or more, consequently it would be very difficult and expensive to compute the Jacobian matrix for each iterate n.

In order to implement the iterative scheme in (5) efficiently it would then be natural to consider another operator for M_n. Axelsson[10] has shown that if M_n is a linear operator and in some sense makes $||L(\phi^n) - M_n\phi^n||$ almost insensitive to ϕ^n, then the iterative procedure (5) converges. In this paper we shall choose M to be an approximation to the full potetntial operator, and with this particular choice the iterative process defines a Newton like algorithm. The construction for M_n is given as follows:

Consider at the nth iterate, the fluid density has been calculated from values of the velocity potential at the (n-1)th iterate, the result of the application of a central difference approximation to L then leads to a nine-point formula, where

$$\begin{aligned}(L\phi)_{i,j} = {} & C_{i,j}\,\phi_{i,j} + W_{i,j}\,\phi_{i-1,j} + E_{i,j}\,\phi_{i+1,j} \\ & + N_{i,j}\,\phi_{i,j+1} + S_{i,j}\,\phi_{i,j-1} + NW_{i,j}\,\phi_{i-1,j+1} \\ & + NE_{i,j}\,\phi_{i+1,j+1} + SW_{i,j}\,\phi_{i-1,j-1} + SE_{i,j}\,\phi_{i+1,j-1}\end{aligned} \tag{6}$$

Note that, the values at the NW, NE, SE and SW positions are usually much smaller than those at N, W, C, E and S positions, since they are due to the skewness effect of the coordinate transformation. The operator M is now chosen by setting the values at NW, NE, SW and SE to zero. Hence M will be a five-point formula and this implies that the skewness effect has been ignored. We would like to remark that neglecting the non-zero values at NW, NE, SW and SE positions for the full potential equation in non-conservation form will no longer quarantee that M will be a good approximation to the L operator. Thus M should retain the nine-point formula structure for this situation, and the results for the non-conservative full potential equation will be reported later.

It should be noted that for the conservative full potential equation M will be identical to the linearized operator L provided an orthogonal transformation is used, and for other transformations M will only be an approximation to L. Thus the operator M takes the following form for orthogonal transformations:

$$M\delta\phi_{i,j} = [\overleftarrow{\partial}_{\xi}\left(\frac{\rho A_1}{J}\right)_{i+1/2,j}\overrightarrow{\partial}_{\xi} + \overleftarrow{\partial}_{\eta}\left(\frac{\rho A_3}{J}\right)_{i,j+1/2}\overrightarrow{\partial}_{\eta}]\delta\phi_{i,j} \qquad (7)$$

in purely subsonic flow calculations. For mixed subsonic-supersonic flow problems, the density ρ in M has been modified according to (3) so that an artificial viscosity term is introduced. However, for large supersonic regions (i.e. strong shock calculations) it is necessary to introduce an additional upwind directional bias in the supersonic flow fields to ensure for smooth convergence. This can be achieved by modifying the

operator M so that a $\phi_{\xi t}$ type term is explicitly included, and M will take the form:

$$M\delta\phi_{i,j} = [\pm \mu\beta \overrightarrow{\partial}_{\xi} \overleftarrow{\partial}_{\xi} (\frac{\rho A_1}{J})_{i+1/2,j} \overrightarrow{\partial}_{\xi} + \overleftarrow{\partial}_{\eta} (\frac{\rho A_3}{J})_{i,j+1/2} \overrightarrow{\partial}_{\eta}]\delta\phi_{i,j} \tag{8}$$

for transonic flow calculations, where μ is a switching function which is zero in subsonic regions and is non-zero in supersonic regions, β is a constant which controls the amount of the $\phi_{\xi t}$ term that is introduced.

It should be noted that Equations (7)-(8) are valid only for orthogonal transformations. In this paper a non-orthogonal grid transformation is used and the operator M will only be an approximation to L. Consequently, instead of Equation (8), we have:

$$M\delta\phi_{i,j} = [\pm\mu\beta \overrightarrow{\partial}_{\xi} \overleftarrow{\partial}_{\xi} + L + E]\,\delta\phi_{i,j} \tag{9}$$

where E is the error matrix due to ignoring the skewness effect in the grid transformation. Now consider $E = E_1 + \alpha I$, then the Newton like iterative scheme in (5) becomes:

$$(\pm\mu\beta \overrightarrow{\partial}_{\xi} \overleftarrow{\partial}_{\xi} + L + \alpha I + E_1)(\phi^{n+1} - \phi^{n}) = -L\phi^{n} \tag{10}$$

Since $||E_1|| << ||L||$ for the full potential equation in conservation form, it is not hard to observe tht Equation (10) in fact simulates a time-dependent problem:

$$\alpha\phi_t \pm \mu\beta\phi_{\xi t} + L\phi^{n+1} = 0 \tag{11}$$

We would also like to remark that our iterative scheme is

fully implicit, and moreover, the boundary conditions for the M operator are the same as that imposed for the full potential operator. Although the iterative scheme given in (5) appears to be similar to that based on the AF technique, there is an important difference in choosing the operator for M. In the AF scheme M is taken to be a product of two simple factors $N_1(\alpha)$ and $N_2(\alpha)$, and the basic iterative scheme can be expressed as:

$$N_1(\alpha)\, N_2(\alpha)\, \delta\phi^n = \alpha\omega L\phi^n \tag{12}$$

where α is a sequence of acceleration parameters, ω is a relaxation parameter, $N_1(\alpha)$ and $N_2(\alpha)$ are both functions of α, and they are easy to invert. The solution of Equation (12) is then obtained in two steps through an intermediate value F, that is

$$\begin{aligned} &\text{step 1:} \quad N_1(\alpha)\, F^n = \alpha\omega L\phi^n \\ &\text{step 2:} \quad N_2(\alpha)\delta\phi^n = F^n \end{aligned} \tag{13}$$

This, in turn, requires an additional boundary conditions for F in order to solve for the step 1. Slow convergence or even divergence may occur if the values of acceleration parameters α and the boundary conditions for F are not carefully chosen. Although the effect of the intermediate boundary conditions in the AF scheme has been studied via the von Neumann and Gustafsson-Kreiss-Sundstrom theory by South and Hafez[11], the stability analysis is valid only for purely subsonic flow calculations and there is still no rigorous analysis available for mixed subsonic-supersonic problems. Consequently, the performance of the AF scheme for transonic flow calculations may

be strongly depended upon the experience of an individual user. That is a fast convergence rate can be achieved if optimal values of acceleration parameters and suitable intermediate boundary conditions are provided, but not otherwise.

Minimal Residual Algorithm

In order to obtain a better approximation ϕ^{n+1} in the Newton like iterative scheme we need to solve a system of linear equations

$$M\,\delta\phi = -\,r \tag{14}$$

where M is a large sparse matrix operator. It is important to have an efficient solution method, since the linear system has to be solved for each step in the Newton like procedure. Direct method would not be possible since it requires a large amount of storages and arithemetic operations. In this paper an iterative method based on a Minimal Residual (MR) algorithm is used. Although the MR method has a slower convergence rate than the Conjugate Gradient algorithm[12], it can be applied to both symmetric and unsymmetric problems as long as all eigenvalues of the matrix operator has positive (or negative) real part. The number of iterations, NI, required to attain a given accuracy ε using the MR method is given[13] by

$$NI = 0.5\ K\ \ln(1/\varepsilon) \tag{15}$$

where $K=||M||\,||M^{-1}||$, is the condition number of the matrix operator M. Clearly the rate of convergence depends upon the value of K, in the sense that the smaller the value for K, the faster is the convergence rate that can be achieved. In order to accelerate the iterative process, a non-singular matrix C is introduced, and the linear system (14) is rewritten as

$$MC^{-1}\,\overline{\delta\phi} = -\,r \qquad (16)$$

where $\overline{\delta\phi} = C\delta\phi$. Equation (16) is known as the preconditioned system and C is the preconditioning operator. Suppose C is chosen so that C^{-1} is a good approximation to M^{-1}, then the condition number of MC^{-1} would be much smaller than that for M itself. Consequently, solving the preconditioned system (16) will yield a faster convergence rate than that for the original system (14). A detailed account of the construction for C and its relationship to the matrix M will be given shortly. The preconditioned MR algorithm can now be described as follows:

Let $\delta\phi^{\circ}$ be an initial guess correction vector, compute the residual vector, $p^{\circ} = -r^{\circ} - M\delta\phi^{\circ}$, and solve $Cz^{\circ} = p^{\circ}$, then for $k=0,1,2,\ldots\overline{k}$, do

$$\begin{aligned}
\delta\phi^{k+1} &= \delta\phi^{k} + \alpha_k z^{k} \quad , \\
p^{k+1} &= p^{k} - \alpha_k Mz^{k} \quad , \\
\text{slove} \quad Cz^{k+1} &= p^{k+1} \quad , \\
\text{where} \quad \alpha_k &= \frac{(p^{k}, Mz^{k})}{(Mz^{k}, Mz^{k})} \quad .
\end{aligned} \qquad (17)$$

Here (x,y) denotes the usual inner product, i.e. $(x,y)=x^{T}y$. The main computational work per iteration in the preconditioned MR algorithm is one matrix-vector multiplication for Mz, and the solution for Cz=p.

Since we are interested in the overall convergence for the nonlinear problem (2), it may not necessary to solve the linear

system in (5) to excessively high accuracy for each Newton like iteration. In our implementation only an approximate solution is sought for each iterate, and this can be achieved by using a small fixed number of iterations (such as $\bar{k}$ is set to 4) in the preconditioned MR algorithm. The iterative procedure described in this section thus consists of outer and inner iteration: the outer iterate is based on a Newton like algorithm (5), and the preconditioned MR method (17) is applied to find an approximate solution for the inner iteration. This method is therefore regarded as a Newton like - Minimal Residual algorithm, or simply an Inexact Newton like (IN) procedure.

Preconditioning Matrix

Having presented the preconditioned MR algorithm (17), we now study how to construct the preconditioning matrix C. Remember that the extra computational work for each iteration in the preconditioned algorithm is in solving the linear system $Cz=p$. Note that, if $C=I$, the identity matrix, then (17) becomes the regular MR algorithm. For a good preconditioned algorithm, C should be chosen so that the condition number of MC^{-1} is much smaller than that of M itself. In particular, if $C=M$ the condition number of MC^{-1} is 1, the minimal value one could obtain. Although a converged solution can be achieved in one iteration, we need to solve $Mz=p$ which is as difficult as solving the original problem $M\delta\phi=-r$. Thus another important consideration is that C^{-1} should be easily invertable, otherwise the preconditioned algorithm will not be efficient. To satisfy these two criteria C is taken to be an approximation to the matrix operator M, and C is also a product of sparse triangular

matrices:

$$C = LU = M + E \tag{18}$$

where L and U are sparse lower and upper triangular matrices and E is the error matrix which measures how good is the approximation between C and M. The matrix C is based on an incomplete factorization technique[14-15], and the algorithm for constructing the sparse matrices L and U can be described as follows. Recall that M is a sparse matrix consisting of five non-zero diagonals, where

$$\begin{aligned}(M\phi)_{i,j} &= E_{i,j}\phi_{i,j} + D_{i,j}\phi_{i-1,j} + F_{i,j}\phi_{i+1,j} \\ &\quad + H_{i,j}\phi_{i,j+1} + B_{i,j}\phi_{i,j-1}\end{aligned} \tag{19}$$

Now L and U are constructed so that L and U has four non-zero diagonals respectively, in which the three non-zero diagonals are in the same positions to those in the lower and upper triangular part of M, where

$$\begin{aligned}(L\phi)_{i,j} &= v_{i,j}\phi_{i,j} + t_{i,j}\phi_{i-1,j} \\ &\quad + g_{i,j}\phi_{i,j-1} + x_{i,j}\phi_{i+1,j-1}\end{aligned} \tag{20}$$

$$\begin{aligned}(U\phi)_{i,j} &= \phi_{i,j} + e_{i,j}\phi_{i+1,j} + f_{i,j}\phi_{i,j+1} \\ &\quad + y_{i,j}\phi_{i-1,j+1}\end{aligned}$$

The elements of L and U are computed from the coefficients of M according to the relations:

$$g_{i,j} = B_{i,j}$$

$$x_{i,j} = -g_{i,j}\, e_{i,j-1}$$

$$
\begin{aligned}
t_{i,j} &= D_{i,j} - g_{i,j}\, y_{i,j-1} \\
v_{i,j} &= (1+\alpha)E_{i,j} - g_{i,j}\, f_{i,j-1} \\
&\quad -t_{i,j}\,(e_{i-1,j} + y_{i-1,j}) \\
&\quad -x_{i,j}\,(y_{i+1,j-1} + e_{i+1,j-1}) \\
e_{i,j} &= (F_{i,j} - f_{i+1,j-1}\, x_{i,j}) / v_{i,j} \\
y_{i,j} &= -t_{i,j}\, f_{i-1,j} / v_{i,j} \\
f_{i,j} &= H_{i,j} / v_{i,j}
\end{aligned}
\qquad (21)
$$

A small value , α, is added to the main diagonal elements of M to ensure the stability of the iterative scheme. The convergence rate, however, is not sensitive to α in the range from 0.025 to 0.1, and α = 0.05 is used for all test problems in this paper. It should be pointed out that the algorithm given in (21) satisfy the following condition[15]:

non-zero elements (except the main diagonal) of M	=	non-zero elements of C which are in the same locations as M

Consequently although the preconditioning matrix C is factorized to LU decomposition, the product of LU will be symmetric as long as M is symmetric. Thus LU may be a symmetric matrix for purely subsonic flows or during the early iterations in mixed subsonic-supersonic calculations. However, when supersonic regions are developed a $\phi_{\xi t}$ term will apperar in the M matrix operator, and LU will then become asymmetric.

The solution of Cp=z can now be obtained efficiently by the

following simple procedure. Since C=LU, the linear system Cz=p can be rewritten as

$$\begin{aligned} Ls &= z \\ Up &= s \end{aligned} \tag{22}$$

where s is a dummy vector. The solution of Ls = z is obtained through a forward substitution, where

$$\begin{aligned} s_{i,j} = (z_{i,j} &- g_{i,j}\, s_{i,j-1} - t_{i,j}\, s_{i-1,j} \\ &- x_{i,j}\, s_{i+1,j-1}) \,/\, v_{i,j} \end{aligned} \tag{23}$$

Note that, unlike the AF scheme, no boundary condition is required in order to solve for the dummy vector s. The solution of Up = s is obtained by a backward substitution, where

$$\begin{aligned} p_{i,j} = s_{i,j} &- e_{i,j}\, p_{i+1,j} - f_{i,j}\, p_{i,j+1} \\ &- y_{i,j}\, p_{i-1,j+1} \end{aligned} \tag{24}$$

To end this section we would like to mention that one of the differences in the implementation of the present method compared to that given in our early refenece[7] is that a more accurate preconditioning matrix is used in this paper. The computational results given in [7] were based on a different LU decomposition, in which $x_{i,j} = y_{i,j} = 0$ for all i,j in Equations (20) (i.e. L and U had only three non-zero diagonals). With an additional diagonal for L and U respectively, the norm of the error matrix E is smaller than that in our early paper. Consequently, a faster convergence rate is achieved in the present iterative scheme.

IV. Numerical Results

In this section results of numerical experiments using AF and IN iterative scheme are presented. The computer program is based on the TAIR code[8], and they all have been carried out on the CDC CYBER 203 computer of NASA Langley Research Center. The problems to be considered are transonic potential flow fields around NACA 0012 airfoil at different Mach numbers and different angles of attack. The grid system used for both schemes is the same, where the total grid is 149*30 = 4470 points. Furthermore, the boundary conditions, initial starting vectors and criterion for convergence test are the same for both schemes. For all figures presented in this sections, the solid lines are the results obtained based on the AF scheme of Holst[8], and the dotted lines are those for the IN iterative scheme.

The surface pressure coefficient distributions, C_p, are identical to those reported by Holst's experiments[8], and hence they will not be presented here. In this paper, we shall mainly focus on the comparison of convergence rates and the efficiency for the AF and IN iterative schemes. Figures 1-3 compare the rates of convergence of the two methods for the following cases: (1) $M_\infty = 0.85$, $\alpha = 0°$; (2) $M_\infty = 0.8$, $\alpha = 0.5°$; and (3) $M_\infty = 0.75$, $\alpha = 2°$. The first one is for zero angle of attack, i.e. a non-lifting condition, (2) and (3) correspond to lifting airfoil calculations.

From the comparison of convergence histories we observe that the IN iterative scheme produces generally a smoother reduction in the residual norm, especially for lifting airfoil calculations. Another useful criterion for comparing the

efficiency of each method is to study the development of supersonic points and the circulation as the number of iterations is increased. Figures 4-5 show the development of the number of supersonic points for $M\infty=0.85$, $\alpha=0°$ and $M\infty=0.8$, $\alpha=0.5°$, and Figure 6 gives the development of circulation for $M\infty=0.8$, $\alpha=0.5°$. The results in Figures 4-6 clearly indicate that the number of supersonic points and the circulations are rapidly established for the IN iterative scheme. From Figures 4-5 we observe that at the 2nd iteration supersonic points had already been established in the IN scheme, and moreover, it reached almost 50% of its final value at the 5th iteration, whereas the AF scheme attained its first supersonic points in the 6th and 7th iterations for the cases of $M\infty=0.85$, and $M\infty=0.8$ respectively.

It should be noted that the convergence histories alone do not reveal the complete picture for the comparison between the AF and IN schemes, another important point of concern is the computing time required for each method to obtain a given accuracy. For a detail comparison one should study the exact numbers of arithmetic operations for each computer code. However, this is obviously very difficult to achieve for practical problems, such as the transonic flow fields calculations. To provide a reasonable indication, Table 1 gives the total CPU time in seconds required to reduce an average residual (i.e. $||r||_2$) to less than 10^{-7}. The CPU time per iterate is 0.235 seconds for the AF scheme, and is 0.524 seconds for the IN scheme.

Table 1. Comparison of CPU time for the AF and IN schemes

	$M_\infty = 0.85$ $\alpha = 0°$	$M_\infty = 0.8$ $\alpha = 0.5°$	$M_\infty = 0.75$ $\alpha = 2°$
AF	32.4	35.5	28.2
IN	41.8	44.5	31.3

It should be pointed out that a considerable amount of computational work is needed for evaluating the residual vactor at each iterate, since it is necessary to update the fluid density at each grid point, modify the densities which are in the supersonic regions, and calculate the velocity potentials, ... etc. In fact evaluating these residuals takes more computing time than solving the discrete potential equation using the AF scheme for each iterate. Consequently, although the work per iterate for the IN scheme takes twice the CPU time required by the AF scheme, the total number of iterations is reduced so that the overall computing time needed to attain the same accuracy for the IN scheme is not significantly larger than that based on the AF scheme.

A further improvement in computing time for the IN iterative scheme is possible, since no effort to optimize the present computer code was attempted. One could expect that a larger number of inner iterations (i.e. the value for k in the preconditioned Minimal Residual algorithm) per outer interation (i.e. the value for n in the Newton like iterative process) might result in a smaller number of outer iterations, and similarly, a smaller number of inner iterations per outer iteration might result in a larger number of outer iterations. It is, of course, not clear what values are the best possible for a particular problem. However, a variable for the inner iteration, so that k

is gradually increased as the outer iterations proceed, will probably be a better choice than a constant value for k as used in the present implementation. A criterion for achieving this objective is being investigated.

Finally, it should be mentioned that in assessing these numerical results, one should keep in mind that there is a major difference between these methods. Although the AF scheme requires less computing time, its formulation and application require specialized knowledge and experience of an individual user. In the sense that the performance of the AF scheme may be greatly affected by the choice of the acceleration parameters and also the treatment of the boundary conditions for the intermediate variable. Because of these reasons, many researchers have had a mixed experience with the AF scheme, sometimes finding excellent results, and sometimes finding them disappointing. Although more computing time is needed for the IN iterative scheme, it is easy to program and it does not suffer from the above difficulties.

V. Conclusion

A Newton like - Minimal Residual iterative scheme is presented for the solution of the full potential equation in transonic ranges. The method described here exhibited an attractive property over the Approximate Factorization scheme, namely the uncertainties and difficulties in choosing the acceleration parameters and treatment of boundary conditions for the intermediate variable are eliminated. Consequently, the present method is less problem dependent and also less user dependent as well. Numerical results for transonic airfoil

calculations are promising: the IN method produces generally a smoother reduction in the residual norm, and the number of supersonic points and circulations are rapidly established as the number of iterations is increased. In addition, there is still room for improvement for the present method. Finally, two potential ares of application are as follows:

(1) Transonic Wing Calculations:

It is technically straightforward to extend the present method for numerical solution of a three-dimensional full potential equation. Moreover, the increase of the computational work over a two-dimensional flow calculation is smaller for the present method compared to the corresponding increase for the Approximate Factorization scheme. For a three-dimensional problem, the matrix operator M in the Newton like iterative procedure will be a seven-point formula instead of a five-point as for a two-dimensional problem. However, the main computational work for the inner iteration is comparable to that required for a two-dimensional calculation, since a sparse LU factorization can be obtained with no difficulty. The Approximate Factorization scheme, on the other hand, consists of three-step calculations[16], and it can be expressed as

$$N_1(\alpha)\, N_2(\alpha)\, N_3(\alpha)\, \delta\phi^n = \alpha\omega L\phi^n \qquad (25)$$

The solution of this scheme would now require two additional boundary conditions for the two intermediate variables.

(2) Finite Element Method in Fluid Dynamics Problems:

Since the Approximate Factorization scheme is essentially

based on the alternating direction splitting methods, they will not be applicable since it is no longer possible, in the finite element formations to partition the matrix operator in terms of the usual directional derivatives. The present method, however, does not suffer from this restriction, since an incomplete sparse LU factorization can still be derived for the finite element matrix.

These two areas of applications and other possibilities will be under investigation.

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Figure Captions

1. Comparison of convergence histories (NACA 0012 airfoil, $M\infty=0.85$, $\alpha=0°$)

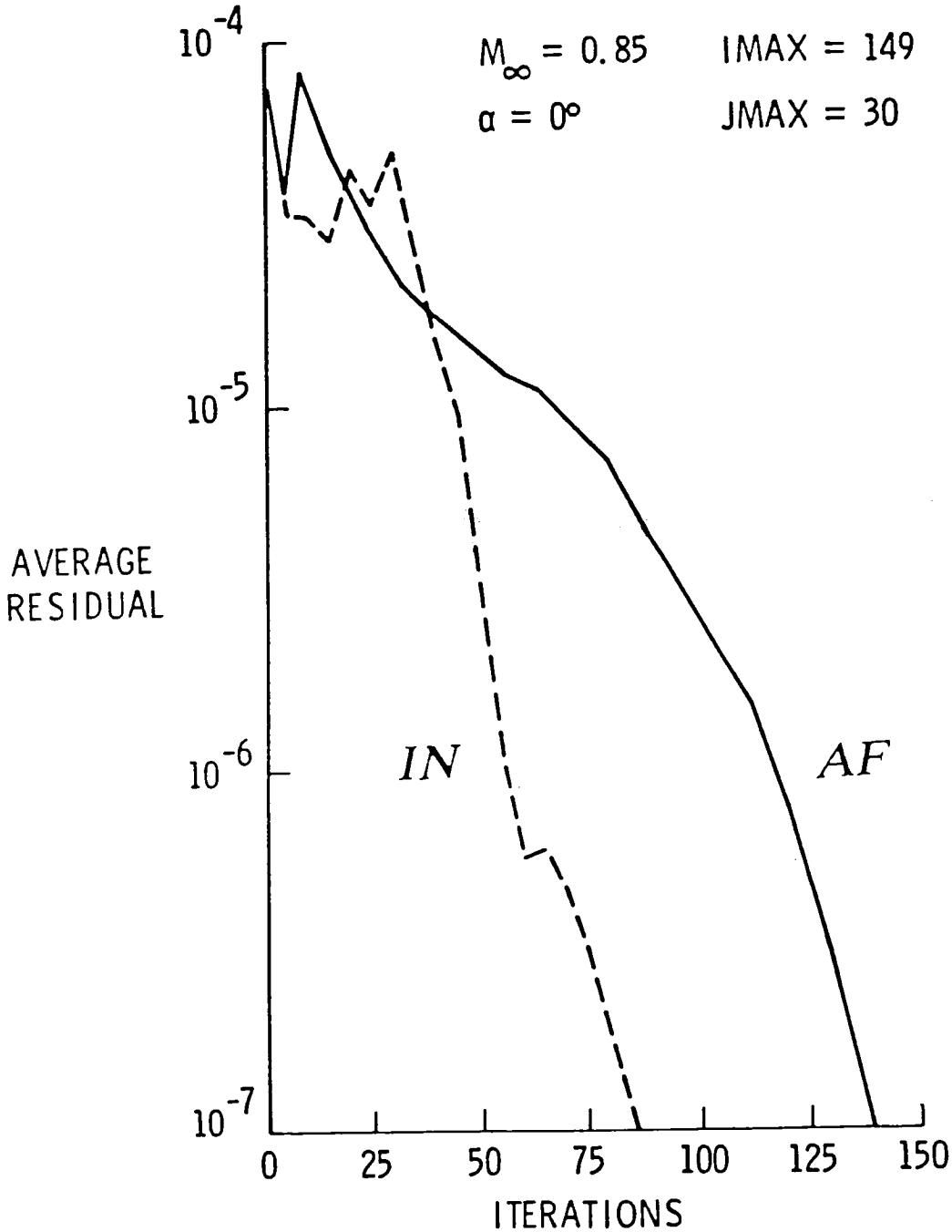
2. Comparison of convergence histories (NACA 0012 airfoil, $M\infty=0.8$, $\alpha=0.5°$)

3. Comparison of convergence histories (NACA 0012 airfoil, $M\infty=0.75$, $\alpha=2°$)

4. Development of the number of supersonic points (NSP) as the number of iterations is increased (NACA 0012 airfoil, $M\infty=0.85$, $\alpha=0°$)

5. Development of the number of supersonic points (NSP) as the number of iterations is increased (NACA 0012 airfoil, $M\infty=0.8$, $\alpha=0.5°$)

6. Development of the circulation (CL) as the number of iterations is increased (NACA 0012 airfoil, $M\infty=0.8$, $\alpha=0.5°$)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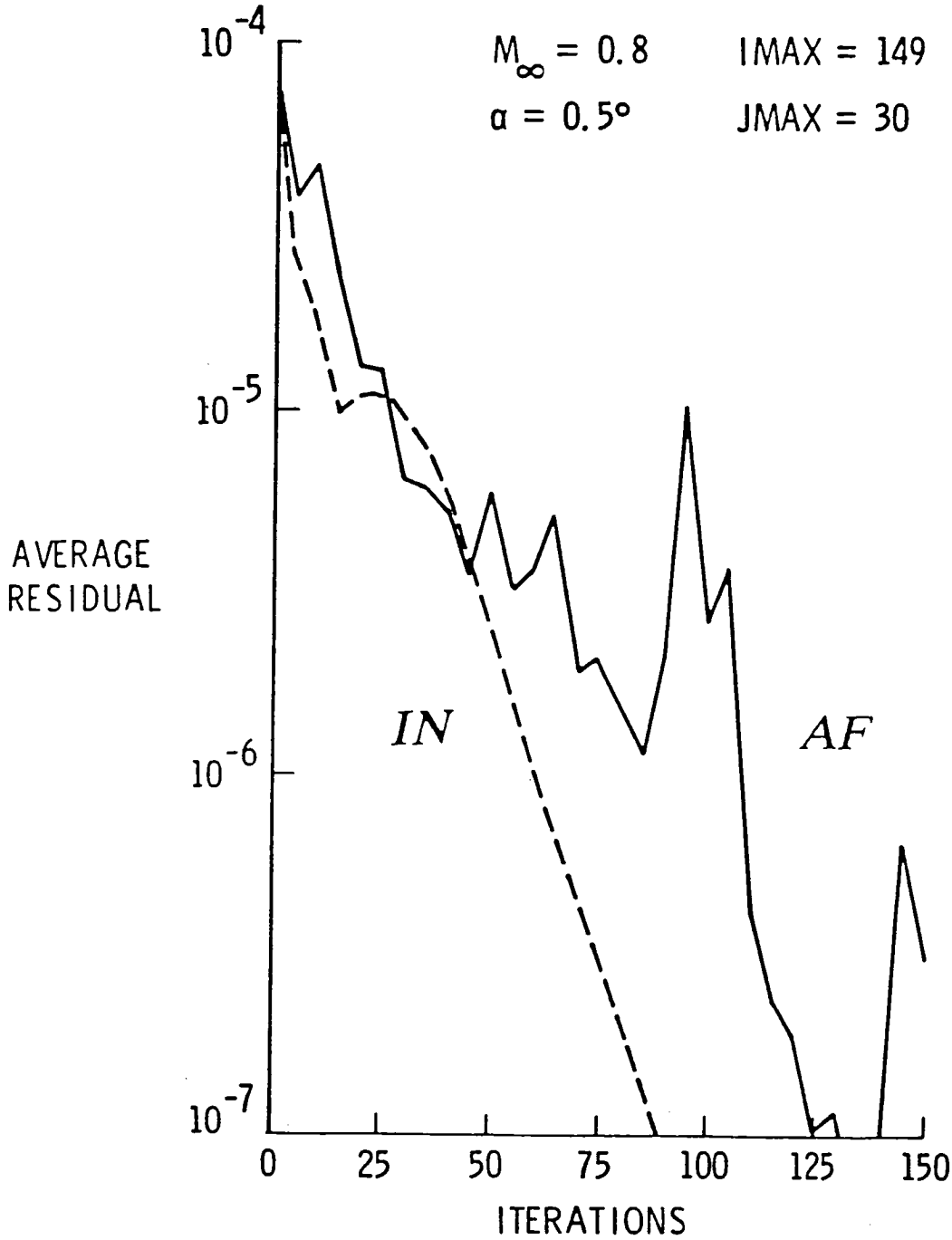
ITERATIONS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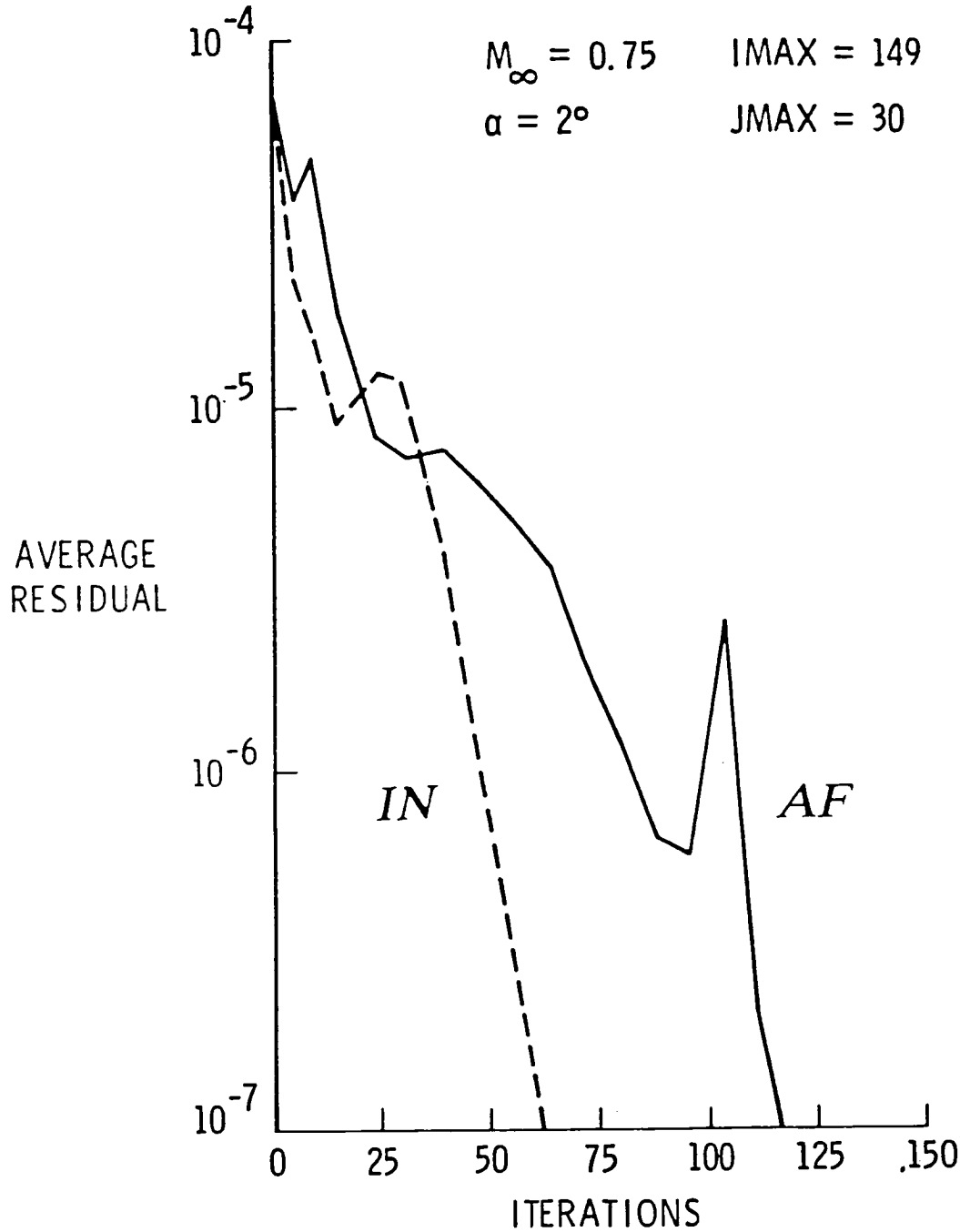
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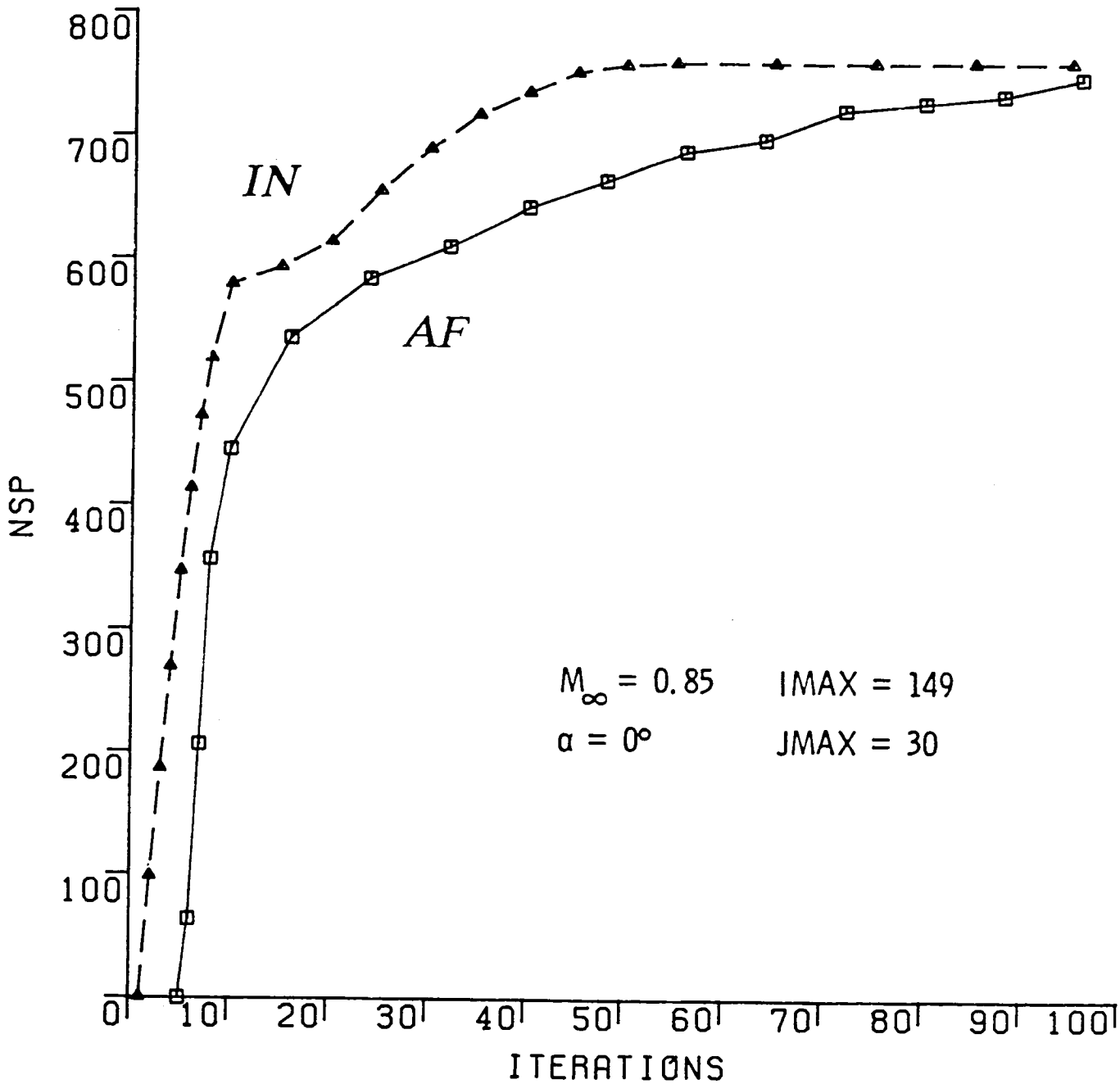
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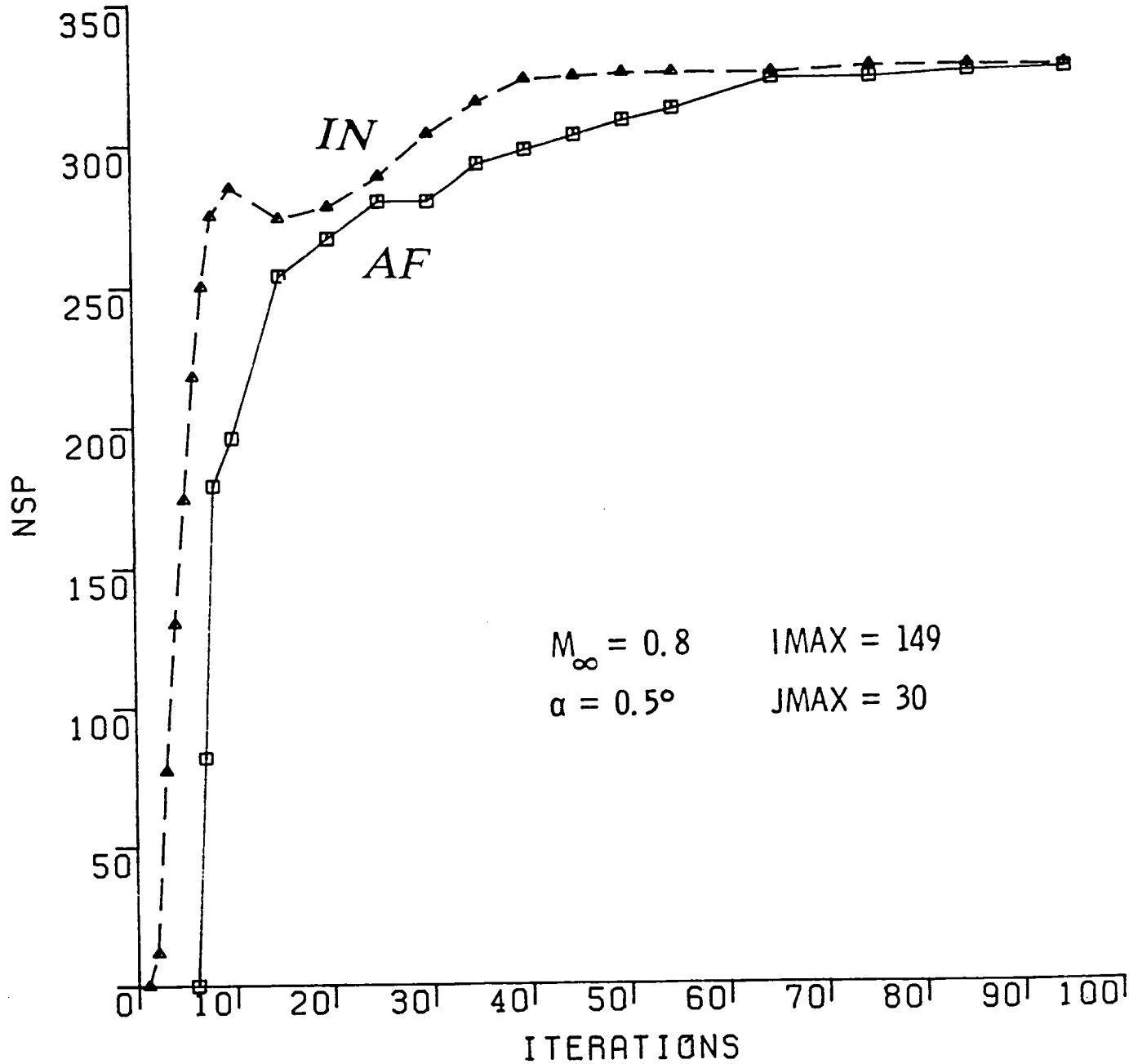
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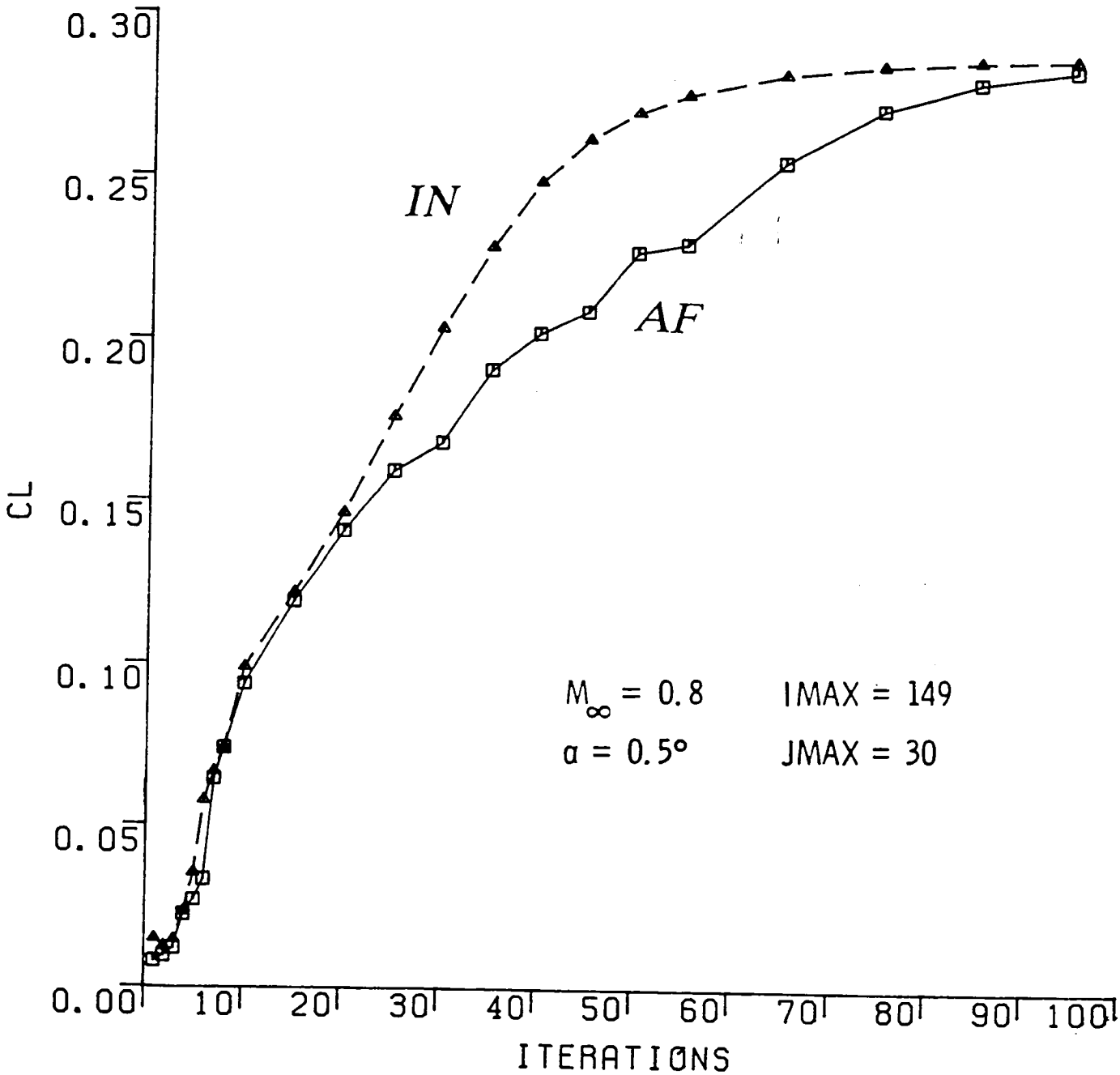
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16. Abstract

A new computational technique for the solution of the full potential equation is presented. The method consists of outer and inner iterations. The outer iterate is based on a Newton like algorithm, and a preconditioned Minimal Residual method is used to seek an approximate solution of the system of linear equations arising at each inner iterate. The present iterative scheme is formulated so that the uncertainties and difficulties associated with many iterative techniques, namely the requirements of acceleration parameters and the treatment of additional boundary conditions for the intermediate variables, are eliminated. Numerical experiments based on the new method for transonic potential flows around NACA 0012 airfoil at different Mach numbers and different angles of attack are presented, and these results are compared with those obtained by the Approximate Factorization technique. Extention to three-dimensional flow calculations and application in finite element methods for fluid dynamics problems by the present method are also discussed.

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